J. Math. Anal. Appl. 377 (2011) 392–405

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Well-posedness for the Navier–Stokes–Nernst–Planck–Poisson system in Triebel–Lizorkin space and Besov space with negative indices ☆

Chao Deng *, Jihong Zhao, Shangbin Cui

Department of Mathematics, Sun Yat-Sen University, Guangzhou, Guangdong 510275, PR China

ARTICLE INFO

Article history:
Received 21 August 2010
Available online 9 November 2010
Submitted by P.G. Lemarie-Rieusset

Keywords:
Navier–Stokes–Nernst–Planck–Poisson system
Mild solutions
Besov space
Triebel–Lizorkin space

ABSTRACT

This paper is concerned with the well-posedness of the Navier–Stokes–Nerst–Planck–Poisson system (NSNPP). Let $s_p = -2 + n/p$. We prove that the NSNPP has a unique local solution $(\vec{u}, v, w) \in \mathcal{E}_{\mathfrak{u}T^*} \times \mathcal{E}_{\mathfrak{v}T^*} \times \mathcal{E}_{\mathfrak{v}T^*}$ for $(\vec{u}_0, v_0, w_0)$ in a subspace, i.e., $\mathcal{V}_{\mathfrak{u}1} \times \mathcal{V}_{\mathfrak{v}1} \times \mathcal{V}_{\mathfrak{v}1}$, of $F_\infty^{-1,2} \times B_p^{s_p,\infty} \times B_p^{s_p,\infty}$ with $\nabla \cdot \vec{u}_0 = 0$. We also prove that there exists a unique small global solution $(\vec{u}, v, w) \in \mathcal{E}_{\mathfrak{u}\infty} \times \mathcal{E}_{\mathfrak{v}\infty} \times \mathcal{E}_{\mathfrak{v}\infty}$ for any small initial data $(\vec{u}_0, v_0, w_0) \in \dot{F}_\infty^{-1,2} \times \dot{B}_p^{s_p,\infty} \times \dot{B}_p^{s_p,\infty}$ with $\nabla \cdot \vec{u}_0 = 0$.

1. Introduction

In this paper, we study the Cauchy problem of the (normalized) n-dimensional Navier–Stokes–Nernst–Planck–Poisson system modeling the motion of an isothermal, incompressible and viscous Newtonian fluid of uniform and homogeneous composition of charged particles (cf. [2,19,21]):

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} + \nabla p = \Delta\vec{u} + \Delta\varphi\nabla\varphi \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.1}$$

$$\nabla \cdot \vec{u} = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.2}$$

$$v_t + \vec{u} \cdot \nabla v = \nabla \cdot (\nabla v - v\nabla\varphi) \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.3}$$

$$w_t + \vec{u} \cdot \nabla w = \nabla \cdot (\nabla w + w\nabla\varphi) \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.4}$$

$$\Delta\varphi = v - w \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.5}$$

$$(\vec{u}, v, w)|_{t=0} = (\vec{u}_0, v_0, w_0) \quad \text{in } \mathbb{R}^n, \tag{1.6}$$

where $\vec{u} = \vec{u}(x,t) \in \mathbb{R}^n$ is the velocity of the fluid, $p = p(x,t) \in \mathbb{R}$ is the pressure in the fluid, $\varphi = \varphi(x,t) \in \mathbb{R}$ is the electrostatic potential caused by the charged particles, $v = v(x,t) \in \mathbb{R}$ and $w = w(x,t) \in \mathbb{R}$ respectively represent the charge densities of the negatively and positively charged particles, and $\vec{u}_0$, v_0 and w_0 are initial data of $\vec{u}$, v and w, respectively. (1.1) and (1.2) are the momentum conservation and the mass conservation equations of the flow, (1.3) and (1.4) are equations modeling the balance between diffusion and convective transport of the charges by the flow and the electric fields, and (1.5) is the Poisson equation for the electrostatic potential φ. Note that the second term on the right-hand side of (1.1)

☆ This work is financially supported by the China National Natural Science Foundation under the Grant No. 10771223.

* Corresponding author.
E-mail addresses: deng315@yahoo.com.cn (C. Deng), zhaojihong2007@yahoo.com.cn (J. Zhao), cuisb3@yahoo.com.cn (S. Cui).

0022-247X/$ – see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2010.11.011

is the Lorentz force caused by the charges. We refer the reader to see [3,2,8,19,20] and the references therein for more details of the physical background of this problem. We also note that for simplicity we have assumed that the fluid density, the viscosity coefficient, the charge mobility and the dielectric constant are all equal to unit. Throughout this paper we assume that the space dimension $n \geqslant 2$ and $s_p = -2 + n/p$.

If the flow is charge-free, i.e., $v = w = \varphi = 0$, then the system (1.1)–(1.5) is essentially the well-known incompressible Navier–Stokes equations (NS). Research of the NS has a long history; we refer the interested reader to see [5–7,10,13–16] and the references cited therein. On the other hand, if we let $\vec{u} \equiv 0$, then (1.1)–(1.5) reduces into the Nernst–Planck–Poisson system (NPP) which was formulated by Nernst and Planck at the end of the nineteenth century as a basic model for the diffusion of ions in an electrolytes (cf. [8]), while in some other literatures it is also called Debye–Hückel system (cf. [4]). Both the Nerst–Planck–Poisson system and the Debye–Hückel system have drawn much attention during the past two decades (cf. [1,3,11] and references cited therein). We note that the coupled system (1.1)–(1.6) not only has a strong physical background, but more interestingly it preserves all the difficulties of NS and NNP, so that it is worthwhile to be taken into deeper considerations. Schmuck in [21] considered global weak solutions of (1.1)–(1.6) under the assumption that $\vec{u}_0 \in L^2(\mathbb{R}^n)$ ($\operatorname{div} \vec{u}_0 = 0$) and $v_0,\ w_0 \in L^\infty(\mathbb{R}^n)$ ($n = 2, 3$), while Ryham in [20] studied the existence, uniqueness and regularity of weak solutions of (1.1)–(1.6) in a bounded domain with no-flux boundary conditions for $L^2(\mathbb{R}^n)$ initial data in two dimensions space and for small initial data in three dimensions space. In our recent work [9], we established well-posedness of (1.1)–(1.6) in modulation spaces. Considering the well-known work of Koch and Tataru [14] on the NS, we naturally wish to establish existence of solutions for initial data $(\vec{u}_0, v_0, w_0)$ in $\dot{F}^{-1,2}_\infty \times \dot{F}^{-1,2}_{L^{n,1}} \times \dot{F}^{-1,2}_{L^{n,1}}$, with $\nabla \cdot \vec{u}_0 = 0$, where $L^{n,1}$ is the Lorenz space. Unfortunately, due to some technical difficulties caused by the nonlinear terms $v\nabla\varphi$ and $w\nabla\varphi$, we are unable to achieve this goal. In this paper we only consider the case that the initial data $\vec{u}_0$, satisfying $\nabla \cdot \vec{u}_0 = 0$, belongs to $\dot{F}^{-1,2}_\infty$, and the initial data (v_0, w_0) belong to a suitable space, which is related to $\dot{B}^{s_p,\infty}_p \times \dot{B}^{s_p,\infty}_p$, i.e., $\mathcal{B}_{\mathrm{vT}}$ (defined below).

We list the scaling property of (1.1)–(1.5) as follows: for any $\lambda > 0$, we denote $\vec{u}_\lambda(x,t) = \lambda\vec{u}(\lambda x, \lambda^2 t)$, $p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$, $v_\lambda(x,t) = \lambda^2 v(\lambda x, \lambda^2 t)$, $w_\lambda(x,t) = \lambda^2 w(\lambda x, \lambda^2 t)$ and $\varphi_\lambda(x,t) = \varphi(\lambda x, \lambda^2 t)$. It is clear that if $(\vec{u}, p, v, w, \varphi)$ is the solution of (1.1)–(1.5) associated with the initial data (1.6), then $(\vec{u}_\lambda, p_\lambda, v_\lambda, w_\lambda, \varphi_\lambda)$ also solves (1.1)–(1.5) with initial data:

$$\vec{u}_{0,\lambda}(x) = \lambda\vec{u}_0(\lambda x), \qquad v_{0,\lambda}(x) = \lambda^2 v_0(\lambda x), \qquad w_{0,\lambda}(x) = \lambda^2 w_0(\lambda x). \tag{1.7}$$

We call the solution to (1.1)–(1.6) a *self-similar solution* if it satisfies the above scaling property. We denote by K_Δ the kernel of $(-\Delta)^{-1}$, then Eq. (1.5) can be rewritten as

$$\varphi(x,t) = (-\Delta)^{-1}(w - v)(x,t) = \int_{\mathbb{R}^n} \big(w(y,t) - v(y,t)\big) K_\Delta(x - y)\,dy. \tag{1.8}$$

In order to treat the incompressible Navier–Stokes equations, it is natural to introduce the Helmholtz projection $\mathbb{P}$, which is a pseudo-differential operator of degree zero and is an orthogonal projection onto the kernel of the divergence operator. Formally, $\mathbb{P}$ is given by the formula $\mathbb{P} = I + \nabla(-\Delta)^{-1}\operatorname{div}$, i.e., $\mathbb{P}$ is the $n \times n$ matrix pseudo-differential operator in $\mathbb{R}^n$ with symbol $(\delta_{jk} - (\xi_j\xi_k)/|\xi|^2)_{j,k=1}^n$, where I represents the unit operator and δ_{jk} is the Kronecker symbol. Making use of this projection operator $\mathbb{P}$ and the heat semigroup $e^{t\Delta}$ (with kernel $K_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-|x|^2/4t}$), we invert the nonlinear system (1.1)–(1.6) into the corresponding integral equations via the Duhamel principle:

$$\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}(\vec{u}\cdot\nabla)\vec{u}\,ds + \int_0^t e^{(t-s)\Delta}\mathbb{P}(\Delta\varphi\nabla\varphi)\,ds, \tag{1.9}$$

$$v = e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta}(\vec{u}\cdot\nabla v)\,ds - \int_0^t e^{(t-s)\Delta}\nabla\cdot(v\nabla\varphi)\,ds, \tag{1.10}$$

$$w = e^{t\Delta}w_0 - \int_0^t e^{(t-s)\Delta}(\vec{u}\cdot\nabla w)\,ds + \int_0^t e^{(t-s)\Delta}\nabla\cdot(w\nabla\varphi)\,ds, \tag{1.11}$$

$$\varphi = (-\Delta)^{-1}(w - v). \tag{1.12}$$

For simplicity of notations, we denote:

$$B_1(\vec{u},\vec{u})(x,t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}(\vec{u}\cdot\nabla)\vec{u}\,ds, \qquad B_2(\varphi,\varphi)(x,t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}(\Delta\varphi\nabla\varphi)\,ds,$$

$$B_3(\vec{u},v)(x,t) = \int_0^t e^{(t-s)\Delta}(\vec{u}\cdot\nabla v)\,ds, \qquad B_4(v,\varphi)(x,t) = \int_0^t e^{(t-s)\Delta}\nabla\cdot(v\nabla\varphi)\,ds.$$

We omit the space time variable if it is not necessary, i.e., denote $B_1(\vec{u},\vec{u})(x,t)$ by $B_1(\vec{u},\vec{u})$, etc. We shall regard the system of Eqs. (1.9)–(1.11) as a fixed point equation for the map $\mathfrak{J}$: $(\vec{u},v,w)\mapsto\mathfrak{J}(\vec{u},v,w)=(\mathfrak{J}_{\mathrm{u}}(\vec{u},v,w),\mathfrak{J}_{\mathrm{v}}(\vec{u},v,w),\mathfrak{J}_{\mathrm{w}}(\vec{u},v,w))$ defined as follows:

$$\mathfrak{J}_{\mathrm{u}}(\vec{u},v,w)=e^{t\Delta}\vec{u}_0-B_1(\vec{u},\vec{u})+B_2(\varphi,\varphi), \tag{1.13}$$

$$\mathfrak{J}_{\mathrm{v}}(\vec{u},v,w)=e^{t\Delta}v_0-B_3(\vec{u},v)-B_4(v,\varphi), \tag{1.14}$$

$$\mathfrak{J}_{\mathrm{w}}(\vec{u},v,w)=e^{t\Delta}w_0-B_3(\vec{u},w)+B_4(w,\varphi), \tag{1.15}$$

where φ is given by (1.12) in the sense of (1.8).

Notations

In this paper, we denote by C, c constants that depend on dimension n and both are greater or equal than 1. $A\lesssim B$ stands for $A\leqslant CB$ and $A\sim B$ stands for $A\lesssim B\lesssim A$. For any $1\leqslant q\leqslant\infty$, we denote $L^q(dx)$, $L^q(dx\,dt)$ by L^q_x, $L^q_{x,t}$, respectively. ∂_i stands for the partial derivative with respect to the ith space variable, i.e., $\partial_i=\partial_{x_i}$. We denote by R_i the Riesz transforms. And throughout this paper, we assume that $\max\{1,\frac{n}{2}\}<p<n$, $\frac{1}{2p_1}+\frac{1}{n}=\frac{1}{p}$, and we will not distinguish the vector valued function space and scalar function space if there is no confusions.

Plan of the paper

This paper is organized as follows: in Section 2 we first introduce the resolution spaces and initial data spaces, then we list the main results of this paper, i.e., Theorem 1 and Corollary 2, while in Section 3 we will prove several propositions of the initial data spaces and we will also list several Lemmas which are needed in Section 4. Section 4 is devoted to the proof of the linear and bilinear estimates. Finally, in Section 5, we will use these estimates obtained in Section 4 to prove Theorem 1.

2. Definitions and the statements of the main result

We start with the definition of the resolution spaces and the initial data spaces. Let $B_{x,r}=\{y\in\mathbb{R}^n: |y-x|<r\}$.

Definition 2.1. (Resolution spaces) Suppose that f and g are measurable functions in $\mathbb{R}^n\times[0,T)$. We write

$$\|f\|_{\mathcal{E}_{\mathrm{uT}}}:=\sup_{t\in(0,T)}t^{\frac{1}{2}}\|f\|_\infty+\sup_{x_0\in\mathbb{R}^n,r^2\in(0,T)}\left(\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_0,r}}|f|^2\,dx\,dt\right)^{\frac{1}{2}},$$

$$\|g\|_{\mathcal{E}_{\mathrm{vT}}}:=\sup_{t\in(0,T)}t^{1-\frac{n}{2p}}\|g\|_p,$$

where $\|f\|_\infty:=\|f\|_{L^\infty_x}$ and $\|g\|_p:=\|g\|_{L^p_x}$. Then the spaces $\mathcal{E}_{\mathrm{uT}}$, $\mathcal{E}_{\mathrm{vT}}$ are defined by

$$\mathcal{E}_{\mathrm{uT}}=\{f\in L^2(0,T;L^2_{loc}): \|f\|_{\mathcal{E}_{\mathrm{uT}}}<\infty\},\qquad \mathcal{E}_{\mathrm{vT}}=\{g\in L^\infty_{loc}(0,T;L^\infty_x): \|g\|_{\mathcal{E}_{\mathrm{vT}}}<\infty\}.$$

(Initial data spaces) Suppose that f_0, g_0 are measurable functions on $\mathbb{R}^n$. We write

$$\|f_0\|_{\mathcal{B}_{\mathrm{uT}}}:=\|e^{t\Delta}f_0\|_{\mathcal{E}_{\mathrm{uT}}},\qquad \|f_0\|_{\mathcal{V}_{\mathrm{uT}}}:=\|f_0\|_{\mathcal{B}_{\mathrm{uT}}},\qquad \|f_0\|_{\mathcal{G}_{\mathrm{uT}}}:=\|f_0\|_{\mathcal{V}_{\mathrm{uT}}},$$

$$\|g_0\|_{\mathcal{B}_{\mathrm{vT}}}:=\|e^{t\Delta}g_0\|_{\mathcal{E}_{\mathrm{vT}}},\qquad \|g_0\|_{\mathcal{V}_{\mathrm{vT}}}:=\|g_0\|_{\mathcal{B}_{\mathrm{vT}}},\qquad \|g_0\|_{\mathcal{G}_{\mathrm{vT}}}:=\|g_0\|_{\mathcal{V}_{\mathrm{vT}}}.$$

Then we define the following function spaces:

$$\mathcal{B}_{\mathrm{uT}}=\{f_0\in S'(\mathbb{R}^n): \|e^{t\Delta}f_0\|_{\mathcal{E}_{\mathrm{uT}}}<\infty\},\qquad \mathcal{B}_{\mathrm{vT}}=\{g_0\in S'(\mathbb{R}^n): \|e^{t\Delta}g_0\|_{\mathcal{E}_{\mathrm{vT}}}<\infty\};$$

$$\mathcal{V}_{\mathrm{uT}}=\Big\{f_0\in\mathcal{B}_{\mathrm{uT}}: \lim_{s\downarrow0}\|e^{t\Delta}f_0\|_{\mathcal{E}_{\mathrm{us}}}=0\Big\},\qquad \mathcal{V}_{\mathrm{vT}}=\Big\{g_0\in\mathcal{B}_{\mathrm{vT}}: \lim_{s\downarrow0}\|e^{t\Delta}g_0\|_{\mathcal{E}_{\mathrm{vs}}}=0\Big\};$$

$$\mathcal{G}_{\mathrm{uT}}=\Big\{f_0\in\mathcal{V}_{\mathrm{uT}}: \lim_{t\downarrow0}e^{t\Delta}f_0=f_0\Big\},\qquad \mathcal{G}_{\mathrm{vT}}=\Big\{g_0\in\mathcal{V}_{\mathrm{vT}}: \lim_{t\downarrow0}e^{t\Delta}g_0=g_0\Big\}.$$

We say that $(f_0,g_0,h_0)\in\mathcal{B}_{\mathrm{uT}}\times\mathcal{B}_{\mathrm{vT}}\times\mathcal{B}_{\mathrm{vT}}$ if $\|(f_0,g_0,h_0)\|^2_{\mathcal{B}_{\mathrm{uT}}\times\mathcal{B}_{\mathrm{vT}}\times\mathcal{B}_{\mathrm{vT}}}=\|f_0\|^2_{\mathcal{B}_{\mathrm{uT}}}+\|g_0\|^2_{\mathcal{B}_{\mathrm{vT}}}+\|h_0\|^2_{\mathcal{B}_{\mathrm{vT}}}<\infty$. Similarly, we define $\mathcal{V}_{\mathrm{uT}}\times\mathcal{V}_{\mathrm{vT}}\times\mathcal{V}_{\mathrm{vT}}$, $\mathcal{G}_{\mathrm{uT}}\times\mathcal{G}_{\mathrm{vT}}\times\mathcal{G}_{\mathrm{vT}}$, $\mathcal{E}_{\mathrm{uT}}\times\mathcal{E}_{\mathrm{vT}}\times\mathcal{E}_{\mathrm{vT}}$ as well as their norms. Moreover, we say that a vector function $\vec{u}=(u_1,u_2,\dots,u_n)\in\mathcal{X}$ if $u_i\in\mathcal{X}$ $(1\leqslant i\leqslant n)$ with $\|\vec{u}\|_{\mathcal{X}}:=(\sum_i\|u_i\|^2_{\mathcal{X}})^{1/2}$, where $\mathcal{X}$ can be any (resolution or initial data) space defined above.

Definition 2.2. We say that the tempered distribution f is in BMO_T if

$$\|f\|_{BMO_T} := \sup_{x\in\mathbb{R}^n, r>0}\left(2r^{-n}\int_{B_{x,r}}\int_0^r \left|t\nabla e^{t^2\Delta} f\right|^2 \frac{dt\,dy}{t}\right)^{\frac{1}{2}}$$

$$= \sup_{x\in\mathbb{R}^n, r>0}\left(r^{-n}\int_{B_{x,r}}\int_0^{r^2} \left|\nabla e^{t\Delta} f\right|^2 dt\,dy\right)^{\frac{1}{2}} < \infty.$$

Moreover, we say that $f \in BMO_T^{-1}$ if

$$\|f\|_{BMO_T^{-1}} = \sup_{x\in\mathbb{R}^n, r>0}\left(r^{-n}\int_{B_{x,r}}\int_0^{r^2} \left|e^{t\Delta} f\right|^2 dt\,dy\right)^{\frac{1}{2}} < \infty.$$

The main result of this paper is as follows:

Theorem 1. *Assume that $T \in (0,\infty]$, $(\vec{u}_0, v_0, w_0) \in \mathcal{B}_{\mathbb{u}\mathrm{T}} \times \mathcal{B}_{\mathbb{v}\mathrm{T}} \times \mathcal{B}_{\mathbb{v}\mathrm{T}}$ with $\nabla\cdot\vec{u}_0 = 0$ and $\|(\vec{u}_0, v_0, w_0)\|_{\mathcal{B}_{\mathbb{u}\mathrm{T}}\times\mathcal{B}_{\mathbb{v}\mathrm{T}}\times\mathcal{B}_{\mathbb{v}\mathrm{T}}}$ small enough. Then there exists a unique small mild solution $(\vec{u}, v, w) \in \mathcal{E}_{\mathbb{u}\mathrm{T}} \times \mathcal{E}_{\mathbb{v}\mathrm{T}} \times \mathcal{E}_{\mathbb{v}\mathrm{T}}$. If $(\vec{u}_0, v_0, w_0) \in \mathcal{V}_{\mathbb{u}1} \times \mathcal{V}_{\mathbb{v}1} \times \mathcal{V}_{\mathbb{v}1}$ and $\nabla\cdot\vec{u}_0 = 0$, then there exist $T^* \in (0,1]$ and a unique local mild solution in $\mathcal{E}_{\mathbb{u}T^*} \times \mathcal{E}_{\mathbb{v}T^*} \times \mathcal{E}_{\mathbb{v}T^*}$; Additionally, if $(\vec{u}_0, v_0, w_0) \in \mathcal{G}_{\mathbb{u}1} \times \mathcal{G}_{\mathbb{v}1} \times \mathcal{G}_{\mathbb{v}1}$ and $\nabla\cdot\vec{u}_0 = 0$, then the local mild solution $(\vec{u}, v, w)$ belongs to $C([0,T^*);\ \mathcal{G}_{\mathbb{u}1} \times \mathcal{G}_{\mathbb{v}1} \times \mathcal{G}_{\mathbb{v}1}) \cap \mathcal{E}_{\mathbb{u}T^*} \times \mathcal{E}_{\mathbb{v}T^*} \times \mathcal{E}_{\mathbb{v}T^*}$.*

Remark 2.1. The above theorem contains two results, i.e., the global existence and the local well-posedness. If we choose $T = \infty$, then we obtain a small global solution. The space $\mathcal{E}_{\mathbb{u}\infty}$ is exactly the space X in [14], and $\mathcal{E}_{\mathbb{u}\mathrm{T}}$ is the localized version of $\mathcal{E}_{\mathbb{u}\infty}$ that is used by several authors, cf. [15,17]. Recall the definitions of bmo^{-1}, vmo^{-1} and gmo^{-1} in [17], we observe that $\mathcal{B}_{\mathbb{u}\mathrm{T}}$ is essentially the space BMO_T^{-1} and when $T = 1$, $\mathcal{B}_{\mathbb{u}1}$, $\mathcal{V}_{\mathbb{u}1}$ are exactly the spaces bmo^{-1}, vmo^{-1}, respectively. There is only a slight difference between $\mathcal{G}_{\mathbb{u}1}$ and gmo^{-1}.

As for uniqueness of the NS, the authors showed that under additional condition $\lim_{t\downarrow 0} t^{\frac{1}{2}-\frac{n}{2p}}\|\vec{u}\|_q = 0$ $(n < q \leqslant \infty)$, the mild solution is unique. This condition is regarded as restriction of behavior on L_x^q norm of solutions near $t = 0$, and this condition was proved to be redundant for uniqueness of mild solutions. Encouraged by this observation, definition of $\mathcal{V}_{\mathbb{v}\mathrm{T}}$ seems to be natural. Since C_0^∞ function is not dense in $\dot{B}_p^{s_p,\infty}$, in order to obtain the time continuity, it is reasonable to introduce the corresponding space $\mathcal{G}_{\mathbb{v}\mathrm{T}}$.

Corollary 2. *Solutions $(\vec{u}, v, w) \in \mathcal{E}_{\mathbb{u}\infty} \times \mathcal{E}_{\mathbb{v}\infty} \times \mathcal{E}_{\mathbb{v}\infty}$, obtained in Theorem 1, are self-similar solutions.*

Since the proof follows from definition of self-similar solution, we omit the details.

3. Properties of initial data spaces

In this short section, we list several basic properties of the initial data spaces which will be used in proving Theorem 1.

Lemma 3.1. *(See [12,15].)* (a) *(Fractional integral theorem) Assume that $0 < s < n$. Then $(-\Delta)^{-\frac{s}{2}}$ is bounded from L_x^p to L_x^q for $1 < p < \frac{n}{s}$ and $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$.* (b) *For any $s \geqslant 0$ and $t > 0$, $(-t\Delta)^{\frac{s}{2}} e^{t\Delta}$ is a convolution operator with kernel $K_t^{(s)}(x) \in L_x^1$, where $K_t^{(s)}$ stands for the sth generalized derivative of heat kernel K_t.*

Lemma 3.2 *(Oseen kernel). (See [15].) For $1 \leqslant j,k \leqslant n$ and $t > 0$, the operator $O_{j,k,t} = \frac{1}{\Delta}\partial_j\partial_k e^{t\Delta}$ is a convolution operator $O_{j,k,t} f = K_{j,k,t} * f$, where the kernel $K_{j,k,t}(x)$ satisfies $K_{j,k,t}(x) = t^{-\frac{n}{2}} K_{j,k}(t^{-\frac{1}{2}}x)$ for a smooth function $K_{j,k}$ such that for all $\alpha \in \mathbb{N}^n$ holds*

$$\left(1+|x|\right)^{n+|\alpha|}\partial^\alpha K_{j,k} \in L_x^\infty.$$

Lemma 3.3 *(Mean value theorem). Let $0 < a < b$. Assume that $K_t(x)$ is the kernel of heat semigroup $e^{t\Delta}$. Then for $t_0 \in [a,b]$ and $t \in [0, b - t_0]$, it follows that:*

$$\left|K_{t_0+t}(x) - K_{t_0}(x)\right| \leqslant c\min\left\{t^{\frac{1}{2}}\left(t_0^{\frac{1}{2}} + |x|\right)^{-n-1}, \left(t_0^{\frac{1}{2}} + |x|\right)^{-n}\right\}. \tag{3.1}$$

Moreover, interpolation of the above inequality, for any $\theta \in (0,1)$, we have

$$\left|K_{t_0+t}(x) - K_{t_0}(x)\right| \leqslant ct^{\frac{\theta}{2}}\left(t_0^{\frac{1}{2}} + |x|\right)^{-n-\theta}. \tag{3.2}$$

Proof. The proof is a direct consequence of the decay property of the heat kernel. Hence we omit the details. □

Lemma 3.4. *$BMO^{-1}_\infty = \dot F^{-1,2}_\infty$, $BMO^{-1}_T = \mathcal{B}_{\mathbb{UT}}$, $B_p^{-2+\frac{n}{p},\infty} = \mathcal{B}_{\mathbb{V}1}$ and $\dot B_p^{-2+\frac{n}{p},\infty} = \mathcal{B}_{\mathbb{V}\infty}$.*

Proof. From the equivalent definitions of Triebel–Lizorkin spaces and Besov spaces in [6, p. 181–183], we have $\mathcal{B}_{\mathbb{V}\infty} = \dot B_p^{s_p,\infty}$. $BMO^{-1}_\infty = \dot F^{-1,2}_\infty$ is a well-known result. It is obvious that $\mathcal{B}_{\mathbb{UT}} \subset BMO^{-1}_T$, hence we only need to show $BMO^{-1}_T \subset \mathcal{B}_{\mathbb{UT}}$. For any $f \in BMO^{-1}_T$, one has

$$\sup_{t\in(0,T)} t^{\frac12}\left\|e^{t\Delta} f\right\|_\infty \lesssim \sup_{x_0\in\mathbb{R}^n, t\in(0,T)} \left(\frac{1}{|B_{x_0,\sqrt t}|}\int_0^t \int_{B_{x_0,\sqrt t}} \left|e^{s\Delta} f\right|^2 ds\,dy\right)^{\frac12}.$$

It follows therefore from Definition 2.1 that

$$\begin{aligned}\|f\|_{\mathcal{B}_{\mathbb{UT}}} &= \sup_{t\in(0,T)} t^{\frac12}\left\|e^{t\Delta} f\right\|_\infty + \sup_{x_0\in\mathbb{R}^n, t\in(0,T)} \left(\frac{1}{|B_{x_0,\sqrt t}|}\int_0^t \int_{B_{x_0,\sqrt t}} \left|e^{s\Delta} f\right|^2 ds\,dy\right)^{\frac12} \\ &\lesssim \sup_{x_0\in\mathbb{R}^n, t\in(0,T)} \left(\frac{1}{|B_{x_0,\sqrt t}|}\int_0^t \int_{B_{x_0,\sqrt t}} \left|e^{s\Delta} f\right|^2 ds\,dy\right)^{\frac12} < \infty,\end{aligned}$$

which shows that $f \in \mathcal{B}_{\mathbb{UT}}$. Finally, making use of Theorem 5.3 in [15] and the obvious fact that $\|e^{\Delta} f\|_p \lesssim \sup_{s\in(0,1)} s^{1-\frac{n}{2p}} \times \|e^{s\Delta} f\|_p$, we have $B_p^{-2+\frac{n}{p},\infty} = \mathcal{B}_{\mathbb{V}1}$. Hence we finish the proof. □

Lemma 3.5. *$\nabla(-\Delta)^{-1}$ is bounded from $\mathcal{B}_{\mathbb{VT}}$ to $\mathcal{B}_{\mathbb{UT}}$.*

Proof. For any $g \in \mathcal{B}_{\mathbb{VT}}$, we deduce from Lemma 3.1(a) and the boundedness of Riesz transforms R_i on L^p_x that

$$\begin{aligned} t^{\frac12}\left\|e^{t\Delta}\nabla(-\Delta)^{-1} g\right\|_\infty &\lesssim t^{\frac12-\frac{n}{4p_1}}\left\|e^{\frac t2\Delta}\nabla(-\Delta)^{-1} g\right\|_{2p_1} \lesssim t^{1-\frac{n}{2p}}\left\|R_i e^{\frac t2\Delta} g\right\|_p \\ &\lesssim t^{1-\frac{n}{2p}}\left\|e^{\frac t2\Delta} g\right\|_p. \end{aligned}\tag{3.3}$$

Let $v = e^{t\Delta} g$. Then it suffices to show the following estimate

$$\begin{aligned} \frac{1}{r^n}\int_0^{r^2}\int_{B_{x_0,r}} \left|e^{t\Delta}\nabla(-\Delta)^{-1} g\right|^2 dt\,dy &\lesssim r^{2-\frac{2n}{p}}\int_0^{r^2}\left(t^{1-\frac{n}{2p}}\left\|R_i(-\Delta)^{-\frac12} v\right\|_{\frac{np}{n-p}}\right)^2 t^{-2+\frac{n}{p}}\,dt \\ &\lesssim r^{2-\frac{2n}{p}}\int_0^{r^2}\left(t^{1-\frac{n}{2p}}\left\|(-\Delta)^{-\frac12} v\right\|_{\frac{np}{n-p}}\right)^2 t^{-2+\frac{n}{p}}\,dt \\ &\lesssim \left(\sup_{t\in(0,r^2)} t^{1-\frac{n}{2p}}\|v\|_p\right)^2 r^{2-\frac{2n}{p}}\int_0^{r^2} t^{-2+\frac{n}{p}}\,dt \\ &\lesssim \left(\sup_{t\in(0,r^2)} t^{1-\frac{n}{2p}}\|v\|_p\right)^2. \end{aligned}\tag{3.4}$$

The desired estimate is a direct consequence of the conclusions of Definition 2.1, (3.3) and (3.4). □

4. Linear and bilinear estimates

Making use of the resolution spaces in Definition 2.1, we have the following linear and bilinear estimates.

Proposition 4.1. *Let $B_1(\vec{u},\vec{u})$, $B_2(\varphi,\varphi)$, $B_3(\vec{u},v)$, $B_4(v,\varphi)$, $B_3(\vec{u},w)$, and $B_4(w,\varphi)$ be defined as in (1.13)–(1.15). Then we have*

$$\left\|\left(e^{t\Delta}\vec{u}_0, e^{t\Delta}v_0, e^{t\Delta}w_0\right)\right\|_{\mathcal{E}_{\mathbb{u}T}\times\mathcal{E}_{\mathbb{v}T}\times\mathcal{E}_{\mathbb{v}T}} \lesssim \left\|(\vec{u}_0, v_0, w_0)\right\|_{\mathcal{B}_{\mathbb{u}T}\times\mathcal{B}_{\mathbb{v}T}\times\mathcal{B}_{\mathbb{v}T}}, \tag{4.1}$$

$$\begin{aligned}&\left\|\left(-B_1(\vec{u},\vec{u})+B_2(\varphi,\varphi), -B_3(\vec{u},v)-B_4(v,\varphi), -B_3(\vec{u},w)+B_4(w,\varphi)\right)\right\|_{\mathcal{E}_{\mathbb{u}T}\times\mathcal{E}_{\mathbb{v}T}\times\mathcal{E}_{\mathbb{v}T}}\\ &\quad\lesssim \left\|(\vec{u},v,w)\right\|^2_{\mathcal{E}_{\mathbb{u}T}\times\mathcal{E}_{\mathbb{v}T}\times\mathcal{E}_{\mathbb{v}T}}.\end{aligned} \tag{4.2}$$

Proof. The proof of (4.1) is trivial. In order to prove (4.2), we need to estimate six terms, i.e., $\|B_1(\vec{u},\vec{u})\|_{\mathcal{E}_{\mathbb{u}T}}$, $\|B_2(\varphi,\varphi)\|_{\mathcal{E}_{\mathbb{u}T}}$, $\|B_3(\vec{u},v)\|_{\mathcal{E}_{\mathbb{v}T}}$, $\|B_4(v,\varphi)\|_{\mathcal{E}_{\mathbb{v}T}}$, $\|B_3(\vec{u},w)\|_{\mathcal{E}_{\mathbb{v}T}}$, and $\|B_4(w,\varphi)\|_{\mathcal{E}_{\mathbb{v}T}}$. In fact, it suffices to estimate the first four terms, since the last two terms are similar to the third and fourth terms.

At first, we consider the term $\|B_1(\vec{u},\vec{u})\|_{\mathcal{E}_{\mathbb{u}T}}$. From Lemmas 3.1 and 3.2 in [14], one has

$$\left\|B_1(\vec{u},\vec{u})\right\|_{\mathcal{E}_{\mathbb{u}T}} \lesssim \|\vec{u}\|^2_{\mathcal{E}_{\mathbb{u}T}}. \tag{4.3}$$

In fact, Lemarié-Rieusset in Chapter 16 of [15] also gave a detailed proof of (4.3).

Next, we consider the term $\|B_2(\varphi,\varphi)\|_{\mathcal{E}_{\mathbb{u}T}}$. We will follow the major steps as in [15]. Since some of the proof related to $\nabla\varphi$ is different from that of $\vec{u}$, and also for readers convenience, we give the detail proof of $\nabla\varphi$ as follows. Recall that $\Delta\varphi\nabla\varphi = \nabla\cdot(\nabla\varphi\otimes\nabla\varphi - |\nabla\varphi|^2 I_{n\times n})$. There are two approaches to estimate its L^∞_x norm. One can directly estimate the term $\Delta\varphi\nabla\varphi$ without writing it into two parts. Or one can resort to the following steps by estimating each part of $\Delta\varphi\nabla\varphi$. Why we choose the latter approach is that we need to split $\Delta\varphi\nabla\varphi$ into two parts when we establish the local $L^2_{t,x}$ estimate. And one can use the following estimates to simplify the proof in the local $L^2_{t,x}$ estimate.

$$\begin{aligned} t^{\frac12}\left\|B_2(\varphi,\varphi)\right\|_\infty &= t^{\frac12}\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot\left(\nabla\varphi\otimes\nabla\varphi - |\nabla\varphi|^2 I_{n\times n}\right)ds\right\|_\infty\\ &\lesssim t^{\frac12}\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(\nabla\varphi\otimes\nabla\varphi)\,ds\right\|_\infty + t^{\frac12}\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot\left(|\nabla\varphi|^2 I_{n\times n}\right)ds\right\|_\infty\\ &:= I_{11}+I_{12}, \end{aligned}$$

where

$$\begin{aligned} I_{11} &= t^{\frac12}\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(\nabla\varphi\otimes\nabla\varphi)\,ds\right\|_\infty,\\ I_{12} &= t^{\frac12}\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot\left(|\nabla\varphi|^2 I_{n\times n}\right)ds\right\|_\infty. \end{aligned}$$

Since Riesz transforms R_i and the projection operator $\mathbb{P}$ are bounded on L^q_x with $q\in(1,\infty)$, the decay property of K_{t-s} yields that

$$\begin{aligned} I_{11}, I_{12} &\lesssim t^{\frac12}\int_0^t (t-s)^{-\frac12-\frac{n}{2p_1}}\left\||\nabla\varphi|^2\right\|_{p_1}ds \lesssim t^{\frac12}\int_0^t (t-s)^{-\frac12-\frac{n}{2p_1}}\|\nabla\varphi\|^2_{2p_1}ds\\ &\lesssim t^{\frac12}\int_0^t (t-s)^{-\frac12-\frac{n}{2p_1}}\left\|(-\Delta)^{-\frac12}(v-w)\right\|^2_{2p_1}ds\\ &\lesssim \Big[\sup_{s\in(0,T)} s^{1-\frac{n}{2p}}\left(\|v\|_p+\|w\|_p\right)\Big]^2\int_0^t t^{\frac12}(t-s)^{-\frac12-\frac{n}{2p_1}}s^{-2+\frac{n}{p}}\,ds\\ &\lesssim \|v\|^2_{\mathcal{E}_{\mathbb{v}T}}+\|w\|^2_{\mathcal{E}_{\mathbb{v}T}}. \end{aligned} \tag{4.4}$$

Then we estimate the local $L^2_{x,t}$ norm. For $0<r^2\leqslant T$, we need to estimate

$$\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_0,r}}\left|\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(\nabla\varphi\otimes\nabla\varphi)\,ds\right|^2 dx\,dt, \tag{4.5}$$

$$\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_0,r}}\left|\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot\big(|\nabla\varphi|^2 I_{n\times n}\big)\,ds\right|^2 dx\,dt. \tag{4.6}$$

It suffices to estimate (4.5), since (4.6) can be treated similarly. Let $\chi_{x_0,r}(x)=1_{B_{x_0,2r}}(x)$, $\chi_{x_0,r}(x)=0$ when $x\in B^c_{x_0,3r}$. We rewrite the bilinear term into three terms

$$\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(\nabla\varphi\otimes\nabla\varphi)\,ds=I_{21}-I_{22}-I_{23},$$

with $I_{2i}(i=1,2,3)$ being defined by

$$I_{21}=\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot\big((1-\chi_{x_0,r})(\nabla\varphi\otimes\nabla\varphi)\big)(s)\,ds,$$

$$I_{22}=\frac{1}{\sqrt{-\Delta}}\mathbb{P}\nabla\cdot\int_0^t e^{(t-s)\Delta}\Delta\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)(s)\,ds,$$

$$I_{23}=\frac{1}{\sqrt{-\Delta}}\mathbb{P}\nabla\cdot\int_0^t e^{t\Delta}\sqrt{-\Delta}\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)(s)\,ds.$$

In order to treat I_{21}, for any $\Phi_t=t^{-n}\Phi(t^{-1}x)\in S(\mathbb{R}^n)$, we recall that the skills of estimating $\frac{1}{|B_{x_0,t}|}\int_{B^c_{x_0,t}}\Phi_t(x_0-y)|f(y)-\Phi_t*f(y)|\,dy$ in harmonic analysis, cf. [12] or [22], is dividing $B^c_{x_0,t}$ into a set of balls $\{B_{x_k,t^*}\}_{|k|\geqslant 1}$ covering $B^c_{x_0,t}$, with $t\sim t^*$ and making use of the decay property and integrability of the kernel Φ_t. By using $|\nabla\varphi\otimes\nabla\varphi|\lesssim|\nabla\varphi|^2$ and the decay property as well as integrability of $\partial_l K_{j,k,t}$ (see Lemma 3.2), one gets

$$\begin{aligned}
\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_0,r}}|I_{21}|^2\,dx\,dt&\lesssim\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_0,r}}\left(\int_0^t\int_{Bx_0,2r^c}\frac{1}{|x-y|^{n+1}}|\nabla\varphi|^2\,dy\,ds\right)^2dx\,dt\\
&\lesssim\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_0,r}}\left(\int_0^{r^2}\int_{\bigcup_{k\in\mathbb{Z}^n,|k|\geqslant 1}B_{x_k,r}}\frac{1}{(|k|r)^{n+1}}|\nabla\varphi|^2\,dy\,ds\right)^2dx\,dt\\
&\lesssim\left(\sum_{k\in\mathbb{Z}^n,|k|\geqslant 1}\frac{1}{|k|^{n+1}}\sup_{x_k\in\mathbb{R}^n}\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_k,r}}|\nabla\varphi|^2\,dy\,ds\right)^2\\
&\lesssim\left[\sup_{x_k\in\mathbb{R}^n}\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_k,r}}|\nabla\varphi|^2\,dy\,ds\right]^2.
\end{aligned}\tag{4.7}$$

Since $\nabla\varphi=\nabla(-\Delta)^{-1}(v-w)$, from Lemma 3.5 and (4.7), we obtain

$$\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_0,r}}|I_{21}|^2\,dx\,dt\lesssim\sup_{s>0}s^{4-\frac{2n}{p}}\big(\|v\|_p^2+\|w\|_p^2\big)^2\lesssim\|v\|^4_{\mathcal{E}_{\forall\mathrm{T}}}+\|w\|^4_{\mathcal{E}_{\forall\mathrm{T}}}. \tag{4.8}$$

Now it is turn to estimate I_{22}. By using the boundedness of the Riesz transforms and $\mathbb{P}$ on L^2_x as well as the maximal $L^2_{x,t}$ regularity, one has

$$\frac{1}{r^n}\int_0^{r^2}\int_{\mathbb{R}^n}|I_{22}|^2\,dx\,dt \lesssim \frac{1}{r^n}\int_0^{r^2}\int_{\mathbb{R}^n}\left|\int_0^t e^{(t-s)\Delta}\Delta\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)\,ds\right|^2 dt\,dx$$
$$\lesssim \frac{1}{r^n}\int_0^{r^2}\int_{\mathbb{R}^n}\left|\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)\right|^2 ds\,dx$$
$$\lesssim \frac{1}{r^n}\int_0^{r^2}\int_{\mathbb{R}^n}\left|\frac{1-e^{-s|\xi|^2}}{|\xi|}\right|^2\big|\mathcal{F}\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)(\xi)\big|^2\,d\xi\,ds.$$

Then by using Plancherel's equality and the fundamental inequality, $(1-e^{-s|\xi|^2})|\xi|^{-1}\lesssim s^{\frac{1}{2}}$, we have

$$\frac{1}{r^n}\int_0^{r^2}\int_{\mathbb{R}^n}s\big|\mathcal{F}\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)(\xi)\big|^2\,d\xi\,ds$$
$$\lesssim \frac{1}{r^n}\int_0^{r^2}\int_{\mathbb{R}^n}s\big|\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big|^2\,dx\,ds$$
$$\lesssim \frac{1}{r^n}\int_0^{r^2}s\int_{B_{x_0,3r}}|\nabla\varphi|^4\,dx\,ds \lesssim \frac{1}{r^n}\int_0^{r^2}s\bigg(\int_{B_{x_0,3r}}|\nabla\varphi|^{2p_1}\,dx\bigg)^{\frac{4}{2p_1}}|B_{x_0,r}|^{1-\frac{2}{p_1}}\,ds$$
$$\lesssim \frac{1}{r^n}\int_0^{r^2}s^{1-4+\frac{2n}{p}}r^{n-\frac{2n}{p_1}}\,ds\Big[\sup_{s\in(0,T)}s^{1-\frac{n}{2p}}\|\nabla\varphi\|_{2p_1}\Big]^4$$
$$\lesssim r^{-\frac{2n}{p_1}}r^{-4+\frac{4n}{p}}\Big[\sup_{s\in(0,T)}s^{1-\frac{n}{2p}}\big(\|v\|_p+\|v\|_p\big)\Big]^4$$
$$\lesssim \|v\|^4_{\mathcal{E}_{\mathrm{vT}}}+\|w\|^4_{\mathcal{E}_{\mathrm{vT}}}.$$

Hence we have

$$\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_0,r}}|I_{22}|^2\,dx\,dt \lesssim \frac{1}{r^n}\int_0^{r^2}\int_{\mathbb{R}^n}|I_{22}|^2\,dx\,dt \lesssim \|v\|^4_{\mathcal{E}_{\mathrm{vT}}}+\|w\|^4_{\mathcal{E}_{\mathrm{vT}}}. \tag{4.9}$$

It remains to estimate the third term I_{23}.

$$\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_0,r}}|I_{23}|^2\,dx\,dt$$
$$\leqslant \frac{1}{r^n}\int_0^{r^2}\int_{\mathbb{R}^n}|I_{23}|^2\,dx\,dt$$
$$=2\,\mathrm{Re}\,\frac{1}{r^n}\int_0^{r^2}\Big\langle e^{2r^2\Delta}\big|\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)(s)\big|,\int_0^s\big|\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)(\tau)\big|\,d\tau\Big\rangle_{L^2_x}ds$$
$$+\,2\,\mathrm{Re}\,\frac{1}{r^n}\int_0^{r^2}\Big\langle\big|\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)(s)\big|,\int_0^s e^{2s\Delta}\big|\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)(\tau)\big|\,d\tau\Big\rangle_{L^2_x}ds,$$

where

$$\int_0^{r^2} \big\langle e^{2r^2\Delta}\big|\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)(s)\big|, \frac{1}{r^n}\int_0^s \big|\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)(\tau)\big|d\tau\big\rangle_{L_x^2} ds$$
$$\lesssim \int_0^{r^2} \big\|e^{2r^2\Delta}\big|\big(\chi_{x_0,r}(\nabla\varphi\otimes\nabla\varphi)\big)(s)\big|\big\|_\infty ds \frac{1}{r^n}\int_0^{r^2}\int_{B_{x_0,r}} \big|\nabla\varphi(\tau)\big|^2 dy\,\tau.$$

Similar to (4.7), we have

$$\int_0^{r^2} \big\|e^{2r^2\Delta}\big|\nabla\varphi(s)\big|^2\big\|_\infty ds \lesssim \sup_{x_k\in\mathbb{R}^n}\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_k,r}} |\nabla\varphi|^2\,dy\,ds, \tag{4.10}$$
$$\int_0^{s} \big\|e^{2s\Delta}\big|\nabla\varphi(\tau)\big|^2\big\|_\infty d\tau \lesssim \sup_{x_k\in\mathbb{R}^n}\frac{1}{s^{\frac{n}{2}}}\int_0^{s}\int_{B_{x_k,\sqrt{s}}} |\nabla\varphi|^2\,dy\,d\tau. \tag{4.11}$$

Therefore, from Lemma 3.5 and (4.10)–(4.11), we have

$$\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_0,r}} |I_{23}|^2\,dx\,dt \lesssim \Bigg[\sup_{x_k\in\mathbb{R}^n}\frac{1}{r^n}\int_0^{r^2}\int_{B_{x_k,r}} |\nabla\varphi|^2\,dy\,ds\Bigg]^2 \lesssim \|v\|^4_{\mathcal{E}_{\mathbb{v}\mathrm{T}}} + \|w\|^4_{\mathcal{E}_{\mathbb{v}\mathrm{T}}}. \tag{4.12}$$

Combining (4.4), (4.8), (4.9) and (4.12), we have

$$\big\|B_2(\varphi,\varphi)\big\|_{\mathcal{E}_{\mathbb{u}\mathrm{T}}} \lesssim \|v\|^2_{\mathbb{v}\mathrm{T}} + \|w\|^2_{\mathbb{v}\mathrm{T}}. \tag{4.13}$$

We now consider the term $\|B_3(\vec{u},v)\|_{\mathcal{E}_{\mathbb{v}\mathrm{T}}}$. By condition (1.2), we observe that

$$\begin{aligned}\big\|B_3(\vec{u},v)\big\|_{\mathcal{E}_{\mathbb{v}\mathrm{T}}} &= \sup_{t\in(0,T)} t^{1-\frac{n}{2p}}\bigg\|\int_0^t e^{(t-s)\Delta}\vec{u}\cdot\nabla v\,ds\bigg\|_p \lesssim \sup_{t\in(0,T)} t^{1-\frac{n}{2p}}\bigg\|\int_0^t e^{(t-s)\Delta}\nabla\cdot(\vec{u}v)\,ds\bigg\|_p\\
&\lesssim \sup_{t\in(0,T)} t^{1-\frac{n}{2p}}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{3}{2}+\frac{n}{2p}}\big(s^{\frac{1}{2}}\|\vec{u}\|_\infty s^{1-\frac{n}{2p}}\|v\|_p\big)\,ds\\
&\lesssim \|\vec{u}\|_{\mathcal{E}_{\mathbb{u}\mathrm{T}}}\|v\|_{\mathcal{E}_{\mathbb{v}\mathrm{T}}}.\end{aligned} \tag{4.14}$$

Finally, we only need to estimate the term $\|B_4(v,\varphi)\|_{\mathcal{E}_{\mathbb{v}\mathrm{T}}}$. From the boundedness of Riesz transforms on $L_x^{2p_1}$ and Lemmas 3.1(a) and 3.5, one gets

$$\begin{aligned}\big\|B_4(v,\varphi)\big\|_{\mathcal{E}_{\mathbb{v}\mathrm{T}}} &= \sup_{t\in(0,T)} t^{1-\frac{n}{2p}}\bigg\|\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(v\nabla(-\Delta)^{-1}(v-w)\big)\,ds\bigg\|_p\\
&\lesssim \sup_{t\in(0,T)} t^{1-\frac{n}{2p}}\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{4p_1}}s^{-2+\frac{n}{p}}s^{1-\frac{n}{2p}}\sum_{i=1}^n\big\|R_i(-\Delta)^{-\frac{1}{2}}(v-w)\big\|_{2p_1}s^{1-\frac{n}{2p}}\|v\|_p\,ds\\
&\lesssim \sup_{t\in(0,T)} t^{1-\frac{n}{2p}}\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{4p_1}}s^{-2+\frac{n}{p}}\,ds\sup_{s\in(0,T)}\big[s^{1-\frac{n}{2p}}\big(\|v\|_p+\|w\|_p\big)s^{1-\frac{n}{2p}}\|v\|_p\big]\\
&\lesssim \|v\|^2_{\mathcal{E}_{\mathbb{v}\mathrm{T}}} + \|w\|_{\mathcal{E}_{\mathbb{v}\mathrm{T}}}\|v\|_{\mathcal{E}_{\mathbb{v}\mathrm{T}}}.\end{aligned} \tag{4.15}$$

Similarly, we have

$$\big\|B_3(\vec{u},w)\big\|_{\mathcal{E}_{\mathbb{v}\mathrm{T}}} + \big\|B_4(w,\varphi)\big\|_{\mathcal{E}_{\mathbb{v}\mathrm{T}}} \lesssim \|\vec{u}\|_{\mathbb{u}\mathrm{T}}\|w\|_{\mathbb{v}\mathrm{T}} + \|w\|^2_{\mathcal{E}_{\mathbb{v}\mathrm{T}}} + \|w\|_{\mathcal{E}_{\mathbb{v}\mathrm{T}}}\|v\|_{\mathcal{E}_{\mathbb{v}\mathrm{T}}}. \tag{4.16}$$

The desired estimate now can be deduced from (4.3) and (4.13)–(4.16). □

5. Proof of Theorem 1

Proof of Theorem 1. From Proposition 4.1 we observe that the map $\mathfrak{J}$ is well defined in $\mathcal{E}_{\mathrm{uT}} \times \mathcal{E}_{\mathrm{vT}} \times \mathcal{E}_{\mathrm{vT}}$, and the contraction property is a direct consequence of Proposition 4.1. Consequently, there exists a mild solution $(\vec{u}, v, w) \in \mathcal{E}_{\mathrm{uT}} \times \mathcal{E}_{\mathrm{vT}} \times \mathcal{E}_{\mathrm{vT}}$ associated with small initial data $(\vec{u}_0, v_0, w_0) \in \mathcal{B}_{\mathrm{uT}} \times \mathcal{B}_{\mathrm{vT}} \times \mathcal{B}_{\mathrm{vT}}$ and $\nabla \cdot \vec{u}_0 = 0$ for any $T > 0$. Proof of the second part of Theorem 1 is easy, we omit it. We still denote T^* by T in this section if there is no confusion. So it remains to prove that $(\vec{u}, v, w)$ belongs to $C([0, T); \mathcal{G}_{\mathrm{u}1} \times \mathcal{G}_{\mathrm{v}1} \times \mathcal{G}_{\mathrm{v}1})$. To this end, we divide the proof into three steps.

We first show that $(\vec{u}, v, w)$ is uniformly bounded in $\mathcal{B}_{\mathrm{u}1} \times \mathcal{B}_{\mathrm{v}1} \times \mathcal{B}_{\mathrm{v}1}$ on $[0, T)$ with $0 < T \leqslant 1$. It suffices to prove the following estimates:

$$\sup_{0<t<T} \left\| \left(e^{t\Delta}\vec{u}_0, e^{t\Delta}v_0, e^{t\Delta}w_0\right) \right\|_{\mathcal{B}_{\mathrm{u}1}\times\mathcal{B}_{\mathrm{v}1}\times\mathcal{B}_{\mathrm{v}1}} \lesssim \left\| (\vec{u}_0, v_0, w_0) \right\|_{\mathcal{B}_{\mathrm{u}1}\times\mathcal{B}_{\mathrm{v}1}\times\mathcal{B}_{\mathrm{v}1}}, \tag{5.1}$$

$$\begin{aligned}&\sup_{0<t<T} \left\| (-B_1(\vec{u},\vec{u})(\cdot,t) + B_2(\varphi,\varphi)(\cdot,t), -B_3(\vec{u},v)(\cdot,t) - B_4(v,\varphi)(\cdot,t), -B_3(\vec{u},w)(\cdot,t)\right.\\ &\quad \left. + B_4(w,\varphi)(\cdot,t)) \right\|_{\mathcal{B}_{\mathrm{u}1}\times\mathcal{B}_{\mathrm{v}1}\times\mathcal{B}_{\mathrm{v}1}} \lesssim \left\| (\vec{u}, v, w) \right\|^2_{\mathcal{E}_{\mathrm{uT}}\times\mathcal{E}_{\mathrm{vT}}\times\mathcal{E}_{\mathrm{vT}}}.\end{aligned} \tag{5.2}$$

It is easy to prove (5.1), hence we only need to show (5.2). It suffices to prove

$$\left\| B_1(\vec{u},\vec{u})(\cdot,t) \right\|^2_{\mathcal{B}_{\mathrm{u}1}} + \left\| B_2(\varphi,\varphi)(\cdot,t) \right\|^2_{\mathcal{B}_{\mathrm{u}1}} \lesssim \left\| (\vec{u}, v, w) \right\|^4_{\mathcal{E}_{\mathrm{uT}}\times\mathcal{E}_{\mathrm{vT}}\times\mathcal{E}_{\mathrm{vT}}},$$
$$\left\| B_3(\vec{u},v)(\cdot,t) \right\|^2_{\mathcal{B}_{\mathrm{v}1}} + \left\| B_4(v,\varphi)(\cdot,t) \right\|^2_{\mathcal{B}_{\mathrm{v}1}} \lesssim \left\| (\vec{u}, v, w) \right\|^4_{\mathcal{E}_{\mathrm{uT}}\times\mathcal{E}_{\mathrm{vT}}\times\mathcal{E}_{\mathrm{vT}}},$$

since $\|B_3(\vec{u}, w)\|^2_{\mathcal{B}_{\mathrm{v}1}} + \|B_4(w, \varphi)\|^2_{\mathcal{B}_{\mathrm{v}1}}$ can be treated similarly. We estimate these terms one by one. From Lemma 3.4, for any fixed t with $t \in (0, T)$, there holds

$$\begin{aligned}\left\| B_1(\vec{u},\vec{u})(\cdot,t) \right\|_{\mathcal{B}_{\mathrm{u}1}} &\lesssim \left\| B_1(\vec{u},\vec{u})(\cdot,t) \right\|_{\mathcal{B}_{\mathrm{u}\infty}} \sim \left\| \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(\vec{u}\otimes\vec{u})\,ds \right\|_{BMO^{-1}_\infty}\\ &\lesssim \left\| \int_0^t e^{(t-s)\Delta}(\vec{u}\otimes\vec{u})\,ds \right\|_{BMO_\infty}\\ &\lesssim \left\| \int_0^t e^{(t-s)\Delta}(\vec{u}\otimes\vec{u})\,ds \right\|_{\infty}\\ &\lesssim \|\vec{u}\|^2_{\mathcal{E}_{\mathrm{uT}}}.\end{aligned} \tag{5.3}$$

Actually, (5.3) is proved in [17]. We rewrite $\Delta\varphi\nabla\varphi$ into two parts, and we need to show

$$\begin{aligned}\left\| B_2(\varphi,\varphi)(\cdot,t) \right\|_{\mathcal{B}_{\mathrm{u}1}} &\lesssim \left\| \int_0^t e^{(t-s)\Delta}(\nabla\varphi\otimes\nabla\varphi)\,ds \right\|_{BMO_\infty} + \left\| \int_0^t e^{(t-s)\Delta}|\nabla\varphi|^2\,ds \right\|_{BMO_\infty}\\ &\lesssim \left\| \int_0^t e^{(t-s)\Delta}\left(|\nabla\varphi|^2 + |\nabla\varphi\otimes\nabla\varphi|\right)ds \right\|_{\infty}\\ &\lesssim \sup_{s\in(0,T)}\left[s^{2-\frac{n}{p}}\|\nabla\varphi\|^2_{2p_1}\right]\int_0^t (t-s)^{-\frac{n}{2p_1}}s^{-2+\frac{n}{p}}\,ds\\ &\lesssim \|v\|^2_{\mathcal{E}_{\mathrm{vT}}} + \|w\|^2_{\mathcal{E}_{\mathrm{vT}}}.\end{aligned} \tag{5.4}$$

From definition of $B_4(v, \varphi)$, Lemma 3.5 and decay property of K_t, we obtain

$$\begin{aligned}\left\| B_4(v,\varphi)(\cdot,t) \right\|_{\mathcal{B}_{\mathrm{v}1}} &= \sup_{t\in(0,T)} t^{1-\frac{n}{2p}}\left\| \int_0^t e^{(t-s)\Delta}\nabla\cdot(v\nabla\varphi)\,ds \right\|_p\\ &\lesssim t^{1-\frac{n}{2p}}\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{4p_1}}s^{-2+\frac{n}{p}}\,ds \sup_{s\in(0,T)}\left[s^{2-\frac{n}{p}}\|\nabla\varphi\|_{2p_1}\|v\|_p\right]\\ &\lesssim \|v\|_{\mathcal{E}_{\mathrm{vT}}}\|w\|_{\mathcal{E}_{\mathrm{vT}}} + \|v\|^2_{\mathcal{E}_{\mathrm{vT}}}.\end{aligned} \tag{5.5}$$

Similarly, we have

$$\begin{aligned}\left\|B_3(\vec{u},v)(\cdot,t)\right\|_{\mathcal{B}_{v1}} &= \sup_{t\in(0,T)} t^{1-\frac{n}{2p}}\left\|\int_0^t e^{(t-s)\Delta}\nabla\cdot(\vec{u}v)\,ds\right\|_p \\ &\lesssim t^{1-\frac{n}{2p}}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{3}{2}+\frac{n}{2p}}\,ds \sup_{s\in(0,T)}\left[s^{\frac{3}{2}-\frac{n}{2p}}\|\vec{u}\|_\infty\|v\|_p\right] \\ &\lesssim \|\vec{u}\|_{\mathcal{E}_{uT}}\|v\|_{\mathcal{E}_{vT}}. \end{aligned} \tag{5.6}$$

Therefore, (5.2) follows from (5.3)–(5.6).

Next we verify that $(\vec{u}(t_0), v(t_0), w(t_0)) \in \mathcal{G}_{u1}\times\mathcal{G}_{v1}\times\mathcal{G}_{v1}$ for $t_0\in[0,T)$. From Definition 2.1 we have $(\vec{u}(t_0), v(t_0), w(t_0)) \in \mathcal{G}_{u1}\times\mathcal{G}_{v1}\times\mathcal{G}_{v1}$ when $t_0=0$. Without loss of generality, we assume that $t_0>0$, then we have

$$\begin{aligned}&\left\|\left(e^{t\Delta}\vec{u}(t_0)-\vec{u}(t_0), e^{t\Delta}v(t_0)-v(t_0), e^{t\Delta}w(t_0)-w(t_0)\right)\right\|_{\mathcal{B}_{u1}\times\mathcal{B}_{v1}\times\mathcal{B}_{v1}} \\ &\leqslant \left\|\left(e^{(t+t_0)\Delta}\vec{u}_0-e^{t_0\Delta}\vec{u}_0, e^{(t+t_0)\Delta}v_0-e^{t_0\Delta}v_0, e^{(t+t_0)\Delta}w_0-e^{t_0\Delta}w_0\right)\right\|_{\mathcal{B}_{u1}\times\mathcal{B}_{v1}\times\mathcal{B}_{v1}} \\ &\quad+\left\|\left(\left(e^{t\Delta}-I\right)B_1(\vec{u},\vec{u})(\cdot,t_0), \left(e^{t\Delta}-I\right)B_3(\vec{u},v)(\cdot,t_0), \left(e^{t\Delta}-I\right)B_3(\vec{u},w)(\cdot,t_0)\right)\right\|_{\mathcal{B}_{u1}\times\mathcal{B}_{v1}\times\mathcal{B}_{v1}} \\ &\quad+\left\|\left(\left(e^{t\Delta}-I\right)B_2(\varphi,\varphi)(\cdot,t_0), \left(e^{t\Delta}-I\right)B_4(v,\varphi)(\cdot,t_0), \left(e^{t\Delta}-I\right)B_4(w,\varphi)(\cdot,t_0)\right)\right\|_{\mathcal{B}_{u1}\times\mathcal{B}_{v1}\times\mathcal{B}_{v1}}.\end{aligned}$$

It is obviously that

$$\begin{aligned}&\left\|\left(e^{(t+t_0)\Delta}\vec{u}_0-e^{t_0\Delta}\vec{u}_0, e^{(t+t_0)\Delta}v_0-e^{t_0\Delta}v_0, e^{(t+t_0)\Delta}w_0-e^{t_0\Delta}w_0\right)\right\|_{\mathcal{B}_{u1}\times\mathcal{B}_{v1}\times\mathcal{B}_{v1}} \\ &\leqslant \left\|\left(e^{t\Delta}\vec{u}_0-\vec{u}_0, e^{t\Delta}v_0-v_0, e^{t\Delta}w_0-w_0\right)\right\|_{\mathcal{B}_{u1}\times\mathcal{B}_{v1}\times\mathcal{B}_{v1}},\end{aligned}$$

which vanishes as t goes to 0. By using Lemma 3.3, similar to (5.3), one has

$$\begin{aligned}\left\|e^{t\Delta}B_1(\vec{u},\vec{u})(\cdot,t_0)-B_1(\vec{u},\vec{u})(\cdot,t_0)\right\|_{\mathcal{B}_{u1}} &\lesssim \left\|\int_0^{t_0}\left(e^{(t+t_0-s)\Delta}-e^{(t_0-s)\Delta}\right)(\vec{u}\otimes\vec{u})\,ds\right\|_\infty \\ &\lesssim t^{\frac{1}{2}}t_0^{-\frac{1}{2}}\|\vec{u}\|^2_{\mathcal{E}_{uT}}.\end{aligned} \tag{5.7}$$

Similarly, from (3.2), we have

$$\begin{aligned}&\left\|e^{t\Delta}B_2(\varphi,\varphi)(\cdot,t_0)-B_2(\varphi,\varphi)(\cdot,t_0)\right\|_{\mathcal{B}_{u1}} \\ &\lesssim \left\|e^{t\Delta}B_2(\varphi,\varphi)(\cdot,t_0)-B_2(\varphi,\varphi)(\cdot,t_0)\right\|_{BMO_\infty^{-1}} \\ &\lesssim \left\|\int_0^{t_0}\left(e^{(t+t_0-s)\Delta}-e^{(t_0-s)\Delta}\right)|\nabla\varphi|^2\,ds\right\|_\infty \\ &\lesssim t^{\frac{\theta}{2}}\int_{B_{x_0,2\sqrt{t}}}\int_0^t \frac{|\nabla\varphi|^2}{((t_0-s)^{\frac{1}{2}}+|y|)^{n+\theta}}\,dy\,ds + t^{\frac{\theta}{2}}\int_{B^c_{x_0,2\sqrt{t}}}\int_0^t \frac{|\nabla\varphi|^2}{((t_0-s)^{\frac{1}{2}}+|y|)^{n+\theta}}\,dy\,ds \\ &=: II_1+II_2.\end{aligned}$$

Similar to (4.7), by using Lemma 3.5, one gets

$$\begin{aligned}II_2 &\lesssim \sup_{x_k\in\mathbb{R}^n} t^{\frac{\theta}{2}}t_0^{-\frac{\theta}{2}}t_0^{-\frac{n}{2}}\int_{B_{x_k,\sqrt{t_0}}}\int_0^{t_0}|\nabla\varphi|^2\,dy\,ds \\ &\lesssim t^{\frac{\theta}{2}}t_0^{-\frac{\theta}{2}}\left[\|v\|^2_{\mathcal{E}_{vT}}+\|w\|^2_{\mathcal{E}_{vT}}\right].\end{aligned} \tag{5.8}$$

We bound the term II_1 by splitting the time interval into two disjoint intervals, and we estimate the two terms separately.

$$II_1 \lesssim t^{\frac{\theta}{2}} \int_{B_{x_0,\sqrt{t_0}}} \int_0^{\frac{t_0}{2}} \frac{|\nabla\varphi|^2}{((t_0-s)^{\frac{1}{2}}+|y|)^{n+\theta}}\,dy\,ds$$
$$+\, t^{\frac{\theta}{2}} \int_{B_{x_0,\sqrt{t_0}}} \int_{\frac{t_0}{2}}^{t_0} \frac{|\nabla\varphi|^2}{((t_0-s)^{\frac{1}{2}}+|y|)^{n+\theta}}\,dy\,ds =: II_{11}+II_{12},$$

where

$$II_{11} \lesssim t^{\frac{\theta}{2}} t_0^{-\frac{\theta}{2}} \sup_{x_0\in\mathbb{R}^n} t_0^{-\frac{n}{2}} \int_{B_{x_0,\sqrt{t_0}}} \int_0^{t_0} |\nabla\varphi|^2\,dy\,ds \lesssim t^{\frac{\theta}{2}} t_0^{-\frac{\theta}{2}} \big[\|v\|^2_{\mathcal{E}_{\mathrm{vT}}}+\|w\|^2_{\mathcal{E}_{\mathrm{vT}}}\big] \tag{5.9}$$

and

$$\begin{aligned} II_{12} &= t^{\frac{\theta}{2}} \int_{B_{x_0,\sqrt{t_0}}} \int_{\frac{t_0}{2}}^{t_0} \frac{|\nabla\varphi|^2}{((t_0-s)^{\frac{1}{2}}+|y|)^{n+\theta}}\,dy\,ds \\ &\lesssim t^{\frac{\theta}{2}} \int_{\frac{t_0}{2}}^{t_0} \bigg(\int_{B_{x_0,\sqrt{t_0}}} |\nabla\varphi|^{2p_1}\,dy\bigg)^{\frac{1}{p_1}} \bigg(\int_{B_{x_0,\sqrt{t_0}}} \frac{1}{((t_0-s)^{\frac{1}{2}}+|y|)^{(n+\theta)p_1'}}\,dy\bigg)^{\frac{1}{p_1'}} ds \\ &\lesssim t^{\frac{\theta}{2}} \int_{\frac{t_0}{2}}^{t_0} s^{-2+\frac{n}{p}} (t_0-s)^{\frac{1}{2p_1'}(-np_1'-\theta p_1'+n)}\,ds \big[\|v\|^2_{\mathcal{E}_{\mathrm{vT}}}+\|w\|^2_{\mathcal{E}_{\mathrm{vT}}}\big] \\ &\lesssim t^{\frac{\theta}{2}} t_0^{-\frac{\theta}{2}} \big[\|v\|^2_{\mathcal{E}_{\mathrm{vT}}}+\|w\|^2_{\mathcal{E}_{\mathrm{vT}}}\big], \end{aligned} \tag{5.10}$$

where $\frac{1}{p_1}+\frac{1}{p_1'}=1$ and $\frac{\theta}{2}+\frac{n}{p}<2$. Indeed, since $p\in(\frac{n}{2},n)$, choose θ sufficiently small such that $\frac{\theta}{2}+\frac{n}{p}<2$. Recall that the heat semigroup $e^{t\Delta}$ in L^q_x with $q\in(1,\infty)$ is an analytic semigroup (cf. [18, p. 74, Theorem 6.13]).

$$\begin{aligned} &\big\|e^{t\Delta}B_3(\vec{u},v)(\cdot,t_0)-B_3(\vec{u},v)(\cdot,t_0)\big\|_{\mathcal{B}_{\mathrm{v}1}} \\ &\quad= \sup_{\tau\in(0,1)} \tau^{1-\frac{n}{2p}} \bigg\|e^{\tau\Delta}\int_0^{t_0}\big(e^{(t+t_0-s)\Delta}-e^{(t_0-s)\Delta}\big)\nabla\cdot(\vec{u}v)\,ds\bigg\|_p \\ &\quad\lesssim \sup_{\tau\in(0,1)} \big\|e^{\tau\Delta}(-\tau\Delta)^{1-\frac{n}{2p}}\big\|_1 \bigg\|\int_0^{t_0}\big(e^{(t+t_0-s)\Delta}-e^{(t_0-s)\Delta}\big)(-\Delta)^{\frac{n}{2p}-\frac{1}{2}}\sum_{i=1}^n R_i(u_iv)\,ds\bigg\|_p \\ &\quad\lesssim \bigg\|\big(e^{t\Delta}-I\big)\int_0^{t_0}(-\Delta)^{\frac{n}{2p}-1}e^{(t_0-s)\Delta}(-\Delta)^{\frac{1}{2}}(\vec{u}v)\,ds\bigg\|_p \\ &\quad\lesssim t^{1-\frac{n}{2p}} \bigg\|\int_0^{t_0} e^{(t_0-s)\Delta}(-\Delta)^{\frac{1}{2}}(\vec{u}v)\,ds\bigg\|_p \\ &\quad\lesssim t^{1-\frac{n}{2p}} \int_0^{t_0}(t_0-s)^{-\frac{1}{2}}s^{-\frac{3}{2}+\frac{n}{2p}}\,ds \sup_{s\in(0,T)} s^{\frac{3}{2}-\frac{n}{2p}}\|\vec{u}v\|_p \\ &\quad\lesssim t^{1-\frac{n}{2p}} t_0^{-1+\frac{n}{2p}} \|\vec{u}\|_{\mathcal{E}_{\mathrm{uT}}}\|v\|_{\mathcal{E}_{\mathrm{vT}}}. \end{aligned} \tag{5.11}$$

Similar to (5.11), we can bound $\|e^{t\Delta}B_3(\vec{u},v)(\cdot,t_0)-B_3(\vec{u},v)(\cdot,t_0)\|_{\mathcal{B}_{\mathrm{v}1}}$ by

$$
\begin{aligned}
&\sup_{\tau\in(0,1)} \tau^{1-\frac{n}{2p}} \left\| e^{\tau\Delta} \int_0^{t_0} \big(e^{(t+t_0-s)\Delta} - e^{(t_0-s)\Delta}\big) \nabla\cdot(\nabla\varphi v)\,ds \right\|_p \\
&\lesssim \left\| \int_0^{t_0} \big(e^{(t+t_0-s)\Delta} - e^{(t_0-s)\Delta}\big)(-\Delta)^{\frac{n}{2p}-\frac{1}{2}}(\nabla\varphi v)\,ds \right\|_p \\
&\sim \left\| \int_0^{t_0} \big(e^{t\Delta} - I\big)(-\Delta)^{\frac{n}{2p}-1} e^{(t_0-s)\Delta}(-\Delta)^{\frac{1}{2}}(\nabla\varphi v)\,ds \right\|_p \\
&\lesssim t^{1-\frac{n}{2p}} \int_0^{t_0} (t_0-s)^{-\frac{n}{2p}} s^{-2+\frac{n}{p}}\,ds \sup_{s\in(0,T)} s^{2-\frac{n}{p}} \|\nabla\varphi v\|_{\frac{pn}{2n-p}} \\
&\lesssim t^{1-\frac{n}{2p}} t_0^{-1+\frac{n}{2p}} \big[\|v\|^2_{\mathcal{E}_{vT}} + \|w\|_{\mathcal{E}_{vT}} \|v\|_{\mathcal{E}_{vT}}\big]. \qquad (5.12)
\end{aligned}
$$

From (5.7)–(5.12), we observe that these terms goes to 0 as t goes to 0.

Finally, we need to show the right-continuity of $(\vec{u}, v, w)$ with respect to time variable since the left-continuity can be proved in the same way. There are two subcases: either $t_0 \in (0,T)$ or $t_0 = 0$. If $t_0 \in (0,T)$, then without loss of generality, we assume that $0 < t \leqslant \delta$. We need to show that the following term goes to 0 as $\delta \to 0+$.

$$
\begin{aligned}
&\big\| \big(\vec{u}(t+t_0) - \vec{u}(t_0), v(t+t_0) - v(t_0), w(t+t_0) - w(t_0)\big) \big\|_{\mathcal{B}_{u1}\times\mathcal{B}_{v1}\times\mathcal{B}_{v1}} \\
&\leqslant \big\| \big(e^{(t+t_0)\Delta} - e^{t_0\Delta}\big)(\vec{u}_0, v_0, w_0) \big\|_{\mathcal{B}_{u1}\times\mathcal{B}_{v1}\times\mathcal{B}_{v1}} + \big\| \big(e^{t\Delta} B_1(\vec{u},\vec{u})(\cdot,t_0) - B_1(\vec{u},\vec{u})(\cdot,t_0), \\
&\quad e^{t\Delta} B_3(\vec{u},v)(\cdot,t_0) - B_3(\vec{u},v)(\cdot,t_0), e^{t\Delta} B_3(\vec{u},w)(\cdot,t_0) - B_3(\vec{u},w)(\cdot,t_0)\big) \big\|_{\mathcal{B}_{u1}\times\mathcal{B}_{v1}\times\mathcal{B}_{v1}} \\
&\quad + \big\| \big(e^{t\Delta} B_2(\varphi,\varphi)(\cdot,t_0) - B_2(\varphi,\varphi)(\cdot,t_0), e^{t\Delta} B_4(v,\varphi)(\cdot,t_0) - B_4(v,\varphi)(\cdot,t_0), \\
&\quad e^{t\Delta} B_4(w,\varphi)(\cdot,t_0) - B_4(w,\varphi)(\cdot,t_0)\big) \big\|_{\mathcal{B}_{u1}\times\mathcal{B}_{v1}\times\mathcal{B}_{v1}} \\
&\quad + \big\| \big(B_1(\vec{u},\vec{u})(\cdot,t_0+t) - B_1(\vec{u},\vec{u})(\cdot,t_0), B_3(\vec{u},v)(\cdot,t_0+t) - B_3(\vec{u},v)(\cdot,t_0), \\
&\quad B_3(\vec{u},w)(\cdot,t_0+t) - B_3(\vec{u},w)(\cdot,t_0)\big) \big\|_{\mathcal{B}_{u1}\times\mathcal{B}_{v1}\times\mathcal{B}_{v1}} \\
&\quad + \big\| \big(B_2(\varphi,\varphi)(\cdot,t_0+t) - B_2(\varphi,\varphi)(\cdot,t_0), B_4(v,\varphi)(\cdot,t_0+t) - B_4(v,\varphi)(\cdot,t_0), \\
&\quad B_4(w,\varphi)(\cdot,t_0+t) - B_4(w,\varphi)(\cdot,t_0)\big) \big\|_{\mathcal{B}_{u1}\times\mathcal{B}_{v1}\times\mathcal{B}_{v1}}.
\end{aligned}
$$

The first three terms are treated before. It remains to estimate the last two terms. Let $T' = t_0 + t$. By using the decay properties of K_t, we have

$$
\begin{aligned}
&\big\| B_1(\vec{u},\vec{u})(\cdot,t_0+t) - B_1(\vec{u},\vec{u})(\cdot,t_0) \big\|_{\mathcal{B}_{u1}} \\
&\lesssim \left\| \int_{t_0}^{t_0+t} e^{(t+t_0-s)\Delta}(\vec{u}\otimes\vec{u})(s)\,ds \right\|_\infty \\
&\lesssim \left\| \int_{\mathbb{R}^n} \int_{t_0}^{t_0+t} \frac{(t_0+t)^{\frac{1}{2}}}{s((t_0+t-s)^{\frac{1}{2}} + |y-\cdot|)^{n+1}}\,ds\,dy \right\|_\infty \sup_{s\in(0,t_0+t)} s\big\|\vec{u}(s)\big\|^2_\infty \\
&\lesssim t^{\frac{1}{2}} t_0^{-\frac{1}{2}} \|\vec{u}\|^2_{\mathcal{E}_{uT'}}, \qquad (5.13)
\end{aligned}
$$

which goes to 0 as δ tends to 0. Proof in Section 4 tells us that $\|B_2(\varphi,\varphi)(\cdot,t_0+t) - B_2(\varphi,\varphi)(\cdot,t_0)\|_{\mathcal{B}_{u1}}$ vanishes as δ tends to 0. In the above estimate, in order to obtain the term $t^{\frac{1}{2}} t_0^{-\frac{1}{2}}$, it suffices to consider two cases, i.e., $t < t_0$ and $t \geqslant t_0$. Using the boundedness of Riesz transforms and Lemma 3.1(b), we have

$$
\begin{aligned}
&\big\| B_3(\vec{u},v)(\cdot,t_0+t) - B_3(\vec{u},v)(\cdot,t_0) \big\|_{\mathcal{B}_{v1}} \\
&\lesssim \sup_{\tau\in(0,1)} \tau^{1-\frac{n}{2p}} \left\| e^{\tau\Delta} \int_{t_0}^{t_0+t} e^{(t+t_0-s)\Delta} \nabla\cdot(\vec{u}v)\,ds \right\|_p
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \left\| \int_{t_0}^{t_0+t} e^{(t+t_0-s)\Delta}(-\Delta)^{-1+\frac{n}{2p}} \nabla\cdot(\vec{u}v)\,ds \right\|_p \\
&\lesssim \int_{t_0}^{t_0+t} (t+t_0-s)^{\frac{1}{2}-\frac{n}{2p}} s^{-\frac{3}{2}+\frac{n}{2p}}\,ds\, \|\vec{u}\|_{\mathcal{E}_{\mathfrak{u}T'}} \|v\|_{\mathcal{E}_{\mathfrak{v}T'}} \\
&\lesssim t^{1-\frac{n}{2p}} t_0^{-1+\frac{n}{2p}} \|\vec{u}\|_{\mathcal{E}_{\mathfrak{u}T'}} \|v\|_{\mathcal{E}_{\mathfrak{v}T'}}, \qquad (5.14)
\end{aligned}
$$

where we use the inequality $s^{-\frac{3}{2}+\frac{n}{2p}} \leqslant t_0^{-1+\frac{n}{2p}} s^{-\frac{1}{2}}$. Similarly, we have

$$\left\| B_4(v,\varphi)(\cdot,t_0+t) - B_4(v,\varphi)(\cdot,t_0)\right\|_{\mathcal{B}_{\mathfrak{v}1}} \lesssim t^{1-\frac{n}{2p}} t_0^{-1+\frac{n}{2p}} \left[\|v\|^2_{\mathcal{E}_{\mathfrak{v}T'}} + \|w\|^2_{\mathcal{E}_{\mathfrak{v}T'}}\right]. \qquad (5.15)$$

The terms in (5.7)–(5.15) vanishes as δ goes to 0.

If $t_0 = 0$, for $t \in (0,\delta]$, it follows from (5.2) that

$$
\begin{aligned}
&\left\|\left(\vec{u}(t)-\vec{u}_0, v(t)-v_0, w(t)-w_0\right)\right\|_{\mathcal{B}_{\mathfrak{u}1}\times\mathcal{B}_{\mathfrak{v}1}\times\mathcal{B}_{\mathfrak{v}1}} \\
&\quad \leqslant c\left\|\left(e^{t\Delta}\vec{u}_0-\vec{u}_0, e^{t\Delta}v_0-v_0, e^{t\Delta}w_0-w_0\right)\right\|_{\mathcal{B}_{\mathfrak{u}1}\times\mathcal{B}_{\mathfrak{v}1}\times\mathcal{B}_{\mathfrak{v}1}} + c\left\|(\vec{u},v,w)\right\|^2_{\mathcal{E}_{\mathfrak{u}\delta}\times\mathcal{E}_{\mathfrak{v}\delta}\times\mathcal{E}_{\mathfrak{v}\delta}}. \qquad (5.16)
\end{aligned}
$$

For small δ, if we have $\|(\vec{u},v,w)\|_{\mathcal{E}_{\mathfrak{u}\delta}\times\mathcal{E}_{\mathfrak{v}\delta}\times\mathcal{E}_{\mathfrak{v}\delta}} \leqslant 2c\|(\vec{u}_0,v_0,w_0)\|_{\mathcal{B}_{\mathfrak{u}\delta}\times\mathcal{B}_{\mathfrak{v}\delta}\times\mathcal{B}_{\mathfrak{v}\delta}}$, then combine with $(\vec{u}_0,v_0,w_0) \in \mathcal{G}_{\mathfrak{u}1}\times\mathcal{G}_{\mathfrak{v}1}\times\mathcal{G}_{\mathfrak{v}1}$, the right-hand side of (5.16) goes to 0 as δ goes to 0. In fact, from (4.1) and (4.2) we can prove that

$$\left\|(\vec{u},v,w)\right\|_{\mathcal{E}_{\mathfrak{u}\delta}\times\mathcal{E}_{\mathfrak{v}\delta}\times\mathcal{E}_{\mathfrak{v}\delta}} \leqslant c\left\|(\vec{u}_0,v_0,w_0)\right\|_{\mathcal{B}_{\mathfrak{u}\delta}\times\mathcal{B}_{\mathfrak{v}\delta}\times\mathcal{B}_{\mathfrak{v}\delta}} + c\left\|(\vec{u},v,w)\right\|^2_{\mathcal{E}_{\mathfrak{u}\delta}\times\mathcal{E}_{\mathfrak{v}\delta}\times\mathcal{E}_{\mathfrak{v}\delta}},$$

hence when δ is small enough, we have $4c^2\|(\vec{u}_0,v_0,w_0)\|_{\mathcal{B}_{\mathfrak{u}\delta}\times\mathcal{B}_{\mathfrak{v}\delta}\times\mathcal{B}_{\mathfrak{v}\delta}} < 1$, and

$$\left\|(\vec{u},v,w)\right\|_{\mathcal{E}_{\mathfrak{u}\delta}\times\mathcal{E}_{\mathfrak{v}\delta}\times\mathcal{E}_{\mathfrak{v}\delta}} \leqslant 2c\left\|(\vec{u}_0,v_0,w_0)\right\|_{\mathcal{B}_{\mathfrak{u}\delta}\times\mathcal{B}_{\mathfrak{v}\delta}\times\mathcal{B}_{\mathfrak{v}\delta}}.$$

Particularly, if we choose $\delta = t$ and from (5.16), then the desired result follows. □

References

[1] N. Ben Abdallah, F. Méhats, N. Vauchelet, A note on the long time behavior for the drift-diffusion-Poisson system, C. R. Math. Acad. Sci. Paris 339 (10) (2004) 683–688.
[2] M. Bazant, T. Squires, Induced-charge electro-kinetic phenomena: theory and microfluidic applications, Phys. Rev. Lett. 92 (6) (2004) 066101.
[3] P. Biler, J. Dolbeault, Long time behavior of solutions to Nernst–Plank and Debye–Hückel drift–diffusion systems, Ann. Henri Poincaré 1 (2000) 461–472.
[4] P. Biler, W. Hebisch, T. Nadzieja, The Debye system: existence and large time behavior of solutions, Nonlinear Anal. TMA 23 (1994) 1189–1209.
[5] J. Bourgain, N. Pavlović, Ill-posedness of the Navier–Stokes equations in a critical space in 3D, J. Funct. Anal. 255 (2008) 2233–2247.
[6] M. Cannone, Harmonic analysis tools for solving the incompressible Navier–Stokes equations, in: S. Friedlander, D. Serre (Eds.), Handbook of Mathematical Fluid Dynamics, vol. 3, Elsevier, 2003.
[7] M. Cannone, Y. Meyer, F. Planchon, Solutions autosimilaires des équations de Navier–Stokes, Séminaire "Équations aux Dérivées Partielles" de l'École polytechnique, Exposé VIII, 1993–1994.
[8] P. Debye, E. Hückel, Zur Theorie der Electrolyte, II, Phys. Z. 24 (1923) 305–325.
[9] C. Deng, J. Zhao, S. Cui, Well-posedness for a dissipative system modeling electrohydrodynamics in modulation spaces, Nonlinear Anal. TMA 73 (2010) 2088–2100.
[10] H. Fujita, T. Kato, On the Navier–Stokes initial value problem I, Arch. Ration. Mech. Anal. 16 (1964) 269–315.
[11] H. Gajewski, K. Gröger, On the basic equations for carrier transport in semiconductors, J. Math. Anal. Appl. 113 (1986) 12–35.
[12] L. Grafakos, Classical and Modern Fourier Analysis, China Machine Press, 2006.
[13] T. Kato, Strong L^p solutions of the Navier–Stokes equations in $\mathbb{R}^m$ with applications to weak solutions, Math. Z. 187 (1984) 471–480.
[14] H. Koch, D. Tataru, Well-posedness for the Navier–Stokes equations, Adv. Math. 157 (2001) 22–35.
[15] P.G. Lemarié-Rieusset, Recent Developments in the Navier–Stokes Problem, Research Notes in Mathematics, Chapman & Hall/CRC, 2002.
[16] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. 63 (1934) 193–248.
[17] H. Miura, Remark on uniqueness of mild solutions to the Navier–Stokes equations, J. Funct. Anal. 218 (2005) 110–129.
[18] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1983.
[19] R.F. Probstein, Physicochemical Hydrodynamics, An Introduction, John Wiley and Sons, INC., 1994.
[20] R.J. Ryham, Existence, uniqueness, regularity and long-term behavior for dissipative systems modeling electrohydrodynamics, arXiv:0910.4973v1.
[21] M. Schmuck, Analysis of the Navier–Stokes–Nernst–Planck–Poisson system, Math. Models Methods Appl. Sci. 19 (6) (2009) 993–1015.
[22] E.M. Stein, Harmonic Analysis, Princeton Univ. Press, Princeton, NJ, 1993.